Perturbation Analyticity and Axiomatic Analyticity. I. Connection of the Landau Singularity Manifold with Källén $\Xi_n(t)$ Manifold and Jost DANAD Manifold*

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For a class of Feynman graphs $\mathcal{G}_n$ (single-loop diagrams *with all internal diagonals*), the p-space perturbation Landau singularity manifolds are shown to be formally of the same structure as the Källén $\Xi_n(t)$ manifolds for the x-space axiomatic primitive domain. The boundary of the Landau manifold is then shown to be the (DANAD)′ manifold. The relationship between the (DANAD)′ and the Jost (DANAD) manifold is a precise generalization of what exists between the F_{kl}' and F_{kl} surfaces of Källén and Wightman. Since the (DANAD)′ defines a natural domain of holomorphy, the axiomatic envelope of holomorphy cannot be expected to be continuable beyond (DANAD)′. This (DANAD)′ result furnishes a specific conjecture to part of the envelope of holomorphy.

1. INTRODUCTION

ONE of the unsolved problems in the study of the analytical properties of the vacuum expectation values of the n-fold products of field operators (in short, the n-point functions) is the determination of the envelope of holomorphy[1] for $n>3$ as a consequence of the postulates of local field theory.[2] For our purpose, we shall only cite specifically the following three axioms: (A*1) Lorentz invariance; (A*2) no negative energy states; (A*3) local commutativity. The problem then breaks up into three stages. It calls for the determination of the boundary of

(i) the *primitive domain* D_n^+, as consequence of A*1 and A*2;

(ii) the *union of the permuted domains* D_n, as consequence of A*1, A*2, A*3;

(iii) the *envelope of holomorphy* $E(D_n)$ after performing the analytic completion over the results one gets in (ii). Its nontriviality is reflected in its sweeping power (cf. Fig. 1).

Our present knowledge on the primitive domain D_n^+ is quite satisfactory for all n. (For convenience to our discussion, this is briefly summarized in Sec. 2.) The permuted domains, while straightforward in principle, have not been fully worked out for $n\geqslant 4$. Needless to say, the envelope of holomorphy, beyond the trivial case $n=2$, is presently known *only* for the case $n=3$, which was the work of Källén and Wightman.[3]

Historically, for $n=3$, the establishment of the major part[4] of the envelope of holomorphy was guided by a knowledge of a certain perturbation singularity domain. More specifically, the following statements hold for the 3-point function domain.

(1) The leading boundary of the primitive domain D_3^+ is given by the F_{kl} surfaces

$$2z_{kl}=r+z_{kk}z_{ll}/r,\quad r>0. \tag{1}$$

(2) The singularity domain of the triangle Feynman graph, when the internal masses are allowed to vary from 0 to ∞, is bounded by the F_{kl}' surfaces

$$2z_{kl}=-r-z_{kk}z_{ll}/r,\quad r>0. \tag{2}$$

(3) The leading boundary of the envelope of holomorphy is precisely given by the F_{kl}' surfaces.

One is evidently struck here by the following two features for $n=3$. (A) There is a mysterious connection between the perturbation boundary and the axiomatic primitive boundary, namely that there exists a nontrivial perturbation boundary which has formally the same structure as the boundary of the primitive domain, and that the only difference lies in a change of sign of certain parameters. (B) There is a mysterious coincidence between the perturbation boundary and the axiomatic envelope of holomorphy, namely that a major part of the envelope of holomorphy is precisely given by *the* perturbation boundary which fulfills the connection (A). It is clear that *the statement* (A) *puts a severe selection on the types of admissible perturbation boundaries.*

The question naturally arises as to whether the above statements (A) and (B) are purely accidental for $n=3$, or if there might actually be grounds for a deeper understanding of a general feature.

In the present series of papers on the connection between the perturbation analyticity and the axiomatic analyticity, we shall establish that there exists a class of Feynman graphs $\{\mathcal{G}_n\}$ such that the above state-

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[1] For basic notions on the analytical functions of several complex variables, see, e.g., A. S. Wightman, Lecture Notes (Les Houches, 1960), in *Relations de Dispersions et Particules Élémentaires* (Hermann & Cie., Paris, 1960), pp. 229–313.

[2] A. S. Wightman, Phys. Rev. **101**, 860 (1956); and in *Colloque sur les Problèmes Mathématique de la Theorie Quantique des Champs* (Lille, 1957).

[3] G. Källén and A. S. Wightman, Kgl. Danske Videnskab. Selskab, Mat. Fys. Skrifter **1**, No. 6 (1958).

[4] For the 3-point function domain, there is another piece of envelope of homomorphy, called the $\mathfrak{F}$ surface (see Ref. 3) which does not seem to be related to any perturbation domain. Presumably, there might also be analogs of the $\mathfrak{F}_3$ surfaces to appear in the higher n-point function domain. Such $\mathfrak{F}_n$ surfaces would again be beyond the reach of perturbation examples. We take the viewpoint here that the essential part of the 3-point function boundary was F_{ij}' (which may be regarded as (DANAD)′ of rank 2) rather than the $\mathfrak{F}$ surface.

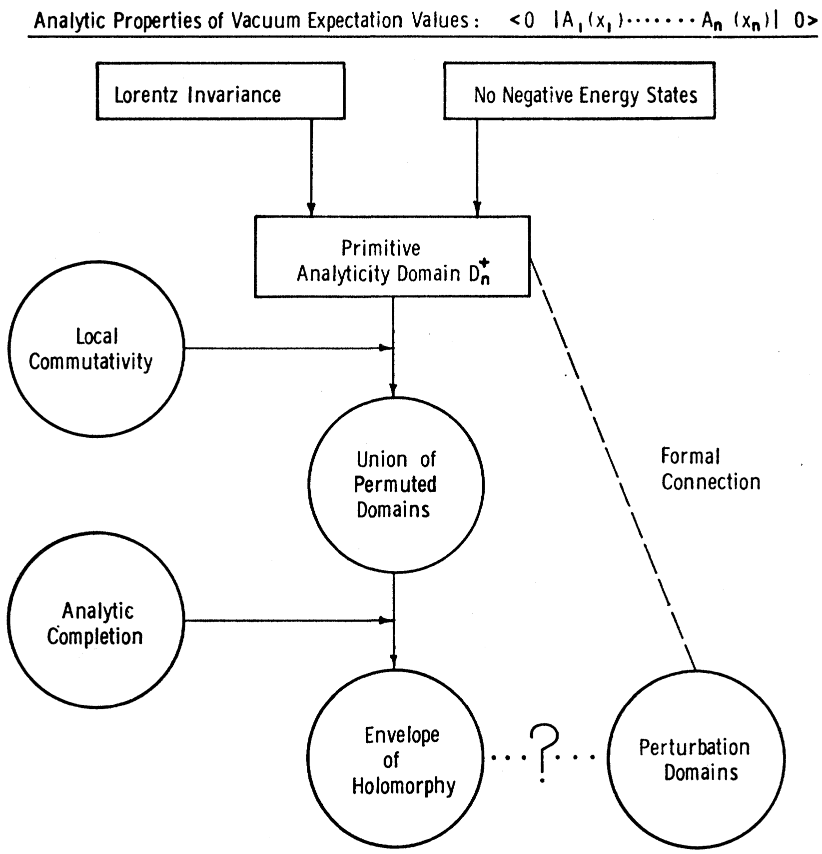

FIG. 1. Schematic summary of the steps involved in the study of the analyticity domains as a consequence of the postulates of local field theory.

ment (A) is valid for all n. The procedures adopted in the present paper will be as follows:

(a) summary of the leading boundary of the primitive domain D_n^+;
(b) determination of the perturbation (Landau) singularity manifolds for $\mathcal{G}_n$;
(c) formal identification of the structure of the Landau singularity manifolds with that of the Källén $\Xi_n(t)$-manifolds[5–7];
(d) determination of the boundary of the Landau singularity manifold through strict analogy with that of the boundary of the $\Xi_n(t)$ manifold;
(e) establishment of statement (A) for all n by direct comparison between (d) and (a).

Note. For $n=4$, the boundary ∂D_4^+ is known as the Jost DANAD manifold,[8,9] parameterized by a set of 3×3 matrices D, A, N having their canonical forms (see Sec. 2). While it is known[10] that for $n\geqslant5$, the matrix N (for rank 4 or higher) no longer has its canonical form of $N_{ik}=1-\delta_{ik}$, nevertheless, for the sake of terminology, we shall still choose to call the leading boundary of D_n^+ as the DANAD manifold parameterized by $(n-1)\times(n-1)$ matrices for general n, with the necessary modification of the form of N being understood.

With this notation, it will be shown that the perturbation boundary for the $\mathcal{G}_n$ justifies the names of the (DANAD)′ manifold. The relationship between the *primed* (DANAD)′ manifold and the *unprimed* DANAD manifold for general n is a precise generalization of that which exists between the F_{kl}' and the F_{kl} surfaces for $n=3$. Of course, the F_{kl}' surfaces may be regarded as the rank-2 (DANAD)′ manifold, *a posteriori*.

This (DANAD)′ result of the perturbation boundary is to be interpreted as follows:

(1) It establishes the desired connection between the perturbation analyticity and the axiomatic analyticity.

(2) It puts a definite upper bound to the axiomatic analyticity domains, namely, the envelope of holomorphy cannot be expected to be continued beyond

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[5] G. Källén and H. Wilhelmsson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Skrifter **1**, No. 9 (1959).
[6] G. Källén, Lecture Notes (Les Houches, 1960) in *Relation de dispersion et particules élémentaries* (Hermann & Cie, Paris, 1960), p. 389.
[7] G. Källén, Nucl. Phys. **25**, 568 (1961).
[8] R. Jost, in *Lectures on Field Theory and Many-Body Problem* (Academic Press Inc., New York, 1961), p. 142.
[9] A. S. Wightman, J. Indian Math. Soc. **24**, 625 (1960).
[10] Strictly speaking, the explicit form of DANAD is meant for cases $n\leqslant4$. For $n\geqslant5$, some peculiarity may arise due to lack of simultaneous normalizability of a set of $(n-1)$ light-like vectors. See, e.g., A. S. Wightman, Ref. 9. For $n=5$, see also A. C. Manoharan, J. Math. Phys. **3**, 853 (1962); and N. H. Möller, Nucl. Phys. **35**, 434 (1962).

TABLE I. Summary of the analyticity of the unpermuted vacuum expectation value (n-point function). It is a remarkable theorem[a] that the approaches from the regular points and from the singular points give the same boundary.

Regular points	Boundary points	Singular points
Wightman tube[b]	Jost DANAD[c]	Källén $\Xi_n(t)$ Manifold[d]

[a] Reference 7.
[b] Reference 2.
[c] References 8 and 9. See note in text near the end of Sec. 1.
[d] References 5, 6, and 7.

(DANAD)′ for configurations such that the (DANAD)′ are relevant.

(3) Finally, if the above statement (B) is to be valid at all for $n>3$, (DANAD)′ would serve as a specific candidate. It is hoped that such a (DANAD)′ knowledge might provide the needed impetus toward constructing an actual proof of part of the envelope of holomorphy.

2. BOUNDARY OF THE PRIMITIVE DOMAIN D_n^+

The primitive analyticity domain of the unpermuted vacuum expectation value

$$W(\zeta_1,\cdots,\zeta_{n-1})\equiv\langle 0|A_1(x_1)\cdots A_n(x_n)|0\rangle, \tag{3}$$

$$\zeta_k\equiv x_{k+1}-x_k, \tag{4}$$

is summarized in Table I. It is known that the boundary can be obtained in two ways:

(a) *Approach to the boundary from the regularity side.* This is done by making use of complex 4 vectors, Lorentz invariance, and the convexity of polyhedral cones. The end result is the Jost DANAD manifold.[8,9] This is a manifold parameterized by the following matrix equations.

$$Z=\mathrm{DANAD}, \tag{5}$$

where all matrices are symmetric, D and N are *real*, Z complex, A complex only along the diagonals. Explicitly [the metric here is (1, 1, 1, −1)],

$$Z_{kl}=-(\zeta_k\cdot\zeta_l)\,,\qquad k,l=1,\cdots,(n-1)\,, \tag{6a}$$

$$D_{kl}=d_k\delta_{kl}\,,\qquad d_k>0\,, \tag{6b}$$

$$\mathrm{Im}A_{kl}=-\epsilon_k\delta_{kl}\,,\qquad \epsilon_k>0\,, \tag{6c}$$

$$N_{ik}=n_{ik}(1-\delta_{ik})\,,\qquad n_{ik}>0\,. \tag{6d}$$

Note. For matrices of rank $r\leqslant 3$, by a scale transformation, N can be normalized to a special form

$$N_{ik}=1-\delta_{ik}. \tag{7}$$

(b) *Approach to the boundary from the singularity side.* This is done with the aid of the singularity manifold of the so-called generalized singular function $\Delta_n^+(z;a)$, which is defined formally as follows[11]:

$$\langle 0|A_1(x_1)\cdots A_n(x_n)|0\rangle =i^{n-1}\int\cdots\int\prod da_{kl}G(a_{kl})\Delta_n^+(z_{kl};a_{kl})\,, \tag{8}$$

with

$$\Delta_n^+(z_{kl};a_{kl})=\left(\frac{i}{2\pi}\right)^{3(n-1)}\int\cdots\int\prod_{k=1}^{n-1}dp_k \times e^{i\Sigma p_j\zeta_j}\prod_k\theta(p_k)\prod_{k\leqslant l}\delta(p_kp_l+a_{kl})\,, \tag{9}$$

where z_{kl} is the same as in (6a) and the "mass" parameters a_{kl} are restricted in a region which is the intersection of the following.

$$(-1)^r\det{}_{(\mathrm{rank}\,r)}|a_{kl}|\leqslant 0\,,\quad 1\leqslant r\leqslant n-1\,. \tag{10}$$

It is known[5,7] that Δ_n^+ $(z;a)$ is an analytic function of z_{kl} except on the following manifold, called the $\Xi_n(t)$ manifold[12]:

$$\Xi_n(t)=\prod(t+\sum_{k=1}^{n-1}\pm\sigma_k^{\frac{1}{2}})=0\,, \tag{11}$$

where t is a real parameter and σ_k are the eigenvalues of the matrix

$$M=Za\,, \tag{12}$$

in which

$$Z=\|z_{kl}\|\,, \tag{12a}$$

$$a=\|a_{kl}\| \tag{12b}$$

are Gram matrices in the x space and p space, respectively. The product in (11) is taken over all distinct sign configurations of $\pm\sigma_k^{\frac{1}{2}}$. Obviously, $\Xi_n(t)$ is a polynomial of degree 2^{n-1} in t. More explicit expressions of $\Xi_n(t)$ for $n\leqslant 8$ as well as the geometrical interpretation of the $\Xi_n(t)$ manifold (for $2\leqslant n<5$) are given in Appendix A.

It has been established by Källén[7] that the abovementioned two approaches are equivalent. Källén has shown that the leading boundary of the $\Xi_n(t)$ manifold is given by the $\frac{1}{2}(n-1)(n-2)$-mass envelopes,[13] and furthermore that such $\frac{1}{2}(n-1)(n-2)$-mass envelopes can (for $n\leqslant 4$) be precisely written in the $Z=$DANAD form. Conversely, it can be easily shown that the DANAD manifold belongs to the $\Xi_n(t)$ manifold.[6,7]

Since an understanding of these statements for the $\Xi_n(t)$ manifold is crucial to our subsequent determination of the boundary of the perturbation Landau singularity manifold, it is perhaps worthwhile to sketch briefly the necessary notions involved here.

[11] Such functions in various forms have been studied by many authors, e.g., A. S. Wightman and D. Hall, Phys. Rev. **99**, 674 (1955), also D. Hall, Ph.D. thesis, Princeton, 1956 (unpublished), for $n\leqslant 3$; for general n, see Refs. 5, 6, and 7. I. Nieminen, Nucl. Phys. **37**, 250 (1962) studied Δ_4^+ in the lower dimensional Lorentz space. Similar results on the Δ_4^+ were independently derived by A. S. Wightman and A. C. T. Wu, 1962 (unpublished), using the technique of integration over the Lorentz group manifold, which technique was first applied by Wightman and Hall, *ibid.*, for $n\leqslant 3$.

[12] It may also read as Im tr$[\pm(Za)^{\frac{1}{2}}]=0$.

[13] Strictly speaking, one is taking simultaneously the geometrical envelopes over t *and* the pertinent number of mass parameters. For simplicity of nomenclature, we shall only count here the number of mass parameters and refer to the envelopes as such.

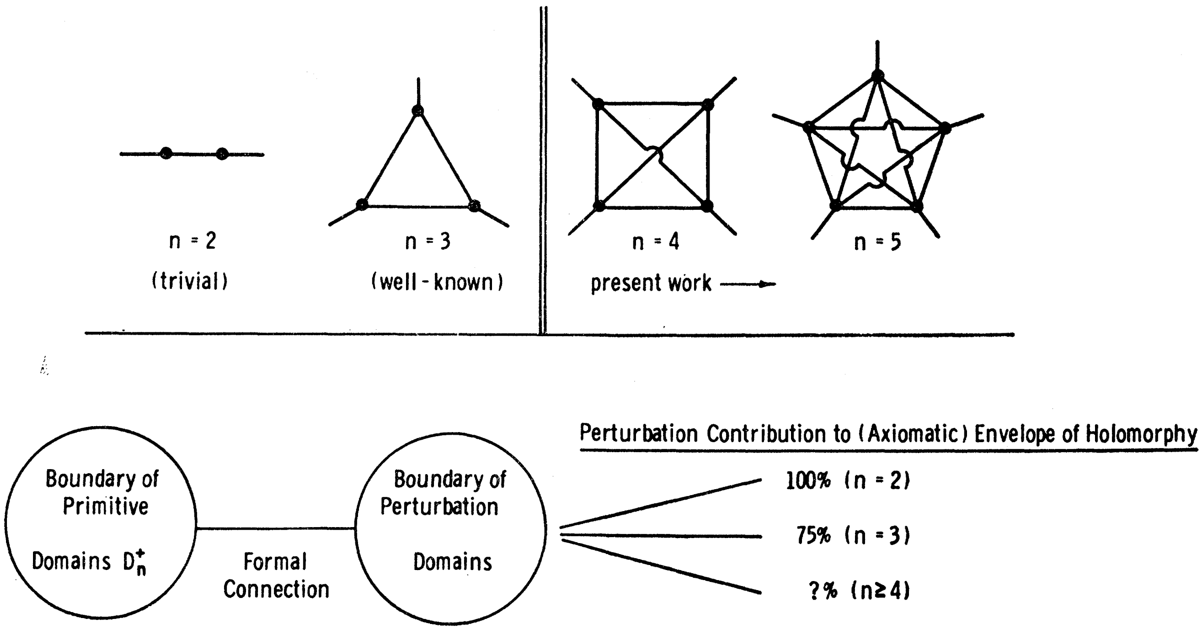

FIG. 2. The class of Feynman graphs $\{\mathcal{G}_n\}$ and the previously known contribution to the envelope of holomorphy for $n \leqslant 3$.

The geometric envelope conditions on the $\Xi_n(t)$ manifold are given by[7]

$$-\partial\Xi_n(t)/\partial t = \omega\,, \tag{13}$$

$$\partial\Xi_n(t)/\partial a_{ik} = \tfrac{1}{2}\omega C_{ik}\,, \tag{14}$$

where ω is just a complex scale factor. The coefficient C_{ik} is *real* if the indices (i,k) belong to the set of nonvanishing parameters a_{ik} over which we are taking the geometric envelopes. Otherwise, $C_{\alpha\beta}$ is complex.[14] Now by a straightforward calculation, the set of Eqs. (11), (13), (14) can be combined to give the geometric envelopes of the $\Xi_n(t)$ manifold. The answer is extremely simple.[7]

$$Z = CaC\,, \tag{15}$$

where Z, a are matrices in (12) and C is the matrix formed from the C_{ik} in (14).

Now there are the following cases:

(1) *Full-mass envelopes.* This implies that all C_{kl} are real, and hence Z real by (15). This is trivial.

(2) *Off-diagonal-mass envelope*[15]. Here, $a_{kk}=0$ and the envelope is taken over all the remaining $a_{kl}, (k\neq l)$. Thus, C_{kk} are complex and C_{kl} real. This is precisely the DANAD manifold, after a trivial scale transformation in (15).[7]

(3) *No-mass envelope.* For our later discussions, it will be convenient to introduce here the notion of the no-mass envelope, or total absence of the geometric envelope conditions. On a purely formal algebraic basis, we can still retain (13), (14) as the defining equations for the slopes. Here, of course, all C_{ik} are complex. The same computation leading to (15) now gives formally the same

$$Z = \mathbf{C}a\mathbf{C} \tag{15a}$$

now, for complex $\mathbf{C}$. It is clear that this is just another way of parameterizing the $\Xi_n(t)$ manifold itself.

With this notion, the result of taking whatever submass geometric envelopes on the $\Xi_n(t)$ manifold is now reduced to a pure substitution scheme into (15a) according to the following prescriptions.

(i) $\mathbf{C}_{kl} = \text{real}\,,$

if envelope condition includes the parameter $a_{kl}(\neq 0)$;

(ii) $\mathbf{C}_{\alpha\beta} = \text{complex}\,,$

if envelope is *not* taken over $a_{\alpha\beta}$. In that case, $a_{\alpha\beta}$ takes on the extreme value, viz., $a_{\alpha\beta}=0$ on the boundary. (16)

We shall find in Sec. 5 that this powerful result[7] on the $\Xi_n(t)$ manifold can be taken over almost word for word for our perturbation Landau singularity manifold.

With the above preliminary, we now come to the discussion of the perturbation singularity.

3. SPECIAL CLASS OF FEYNMAN GRAPHS $\{\mathcal{G}_n\}$

We define $\{\mathcal{G}_n\}$ to be the set of Feynman graphs with n external vertices, zero internal vertices, and $\frac{1}{2}n(n-1)$ internal lines, connecting every pair of vertices once and only once. See Fig. 2. The case $n=2$ is trivial. $n=3$ has been fully discussed.[3,16]

[14] Clearly, there will be no reality restriction on the coefficient $C_{\alpha\beta}$ if the corresponding parameter $a_{\alpha\beta}$ does not enter into the geometric envelope.

[15] In principle, there are of course other intermediate cases where not all diagonal masses are zero, but these are never relevant. See Ref. 7.

[16] A. C. T. Wu, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. 33, No. 3 (1961).

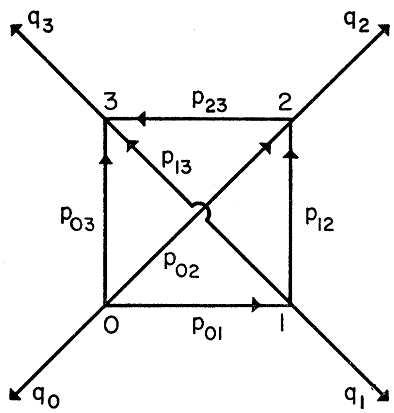

FIG. 3. The graph $\mathcal{G}_4$.

Let the n vertices be labeled by indices $i=0, 1, 2, \cdots, (n-1)$. Let p_{ij} be the 4 momentum directed from vertices i to j. Call $p_{ji}=-p_{ij}$. Let m_{ij} be the mass parameters associated with the line connecting (i,j). To be specific, let us concentrate on the graph $\mathcal{G}_4$. Once the result is known for $n=4$, extension to $n=5$ is immediate. The reason that perhaps one should pause at $n=5$ is that in the 4-dimensional Lorentz space, the maximal rank of the Gram determinent is 4, and we prefer not to get involved at this time with the problem of linear dependence of a set of 4 vectors when $n>5$.

4. LANDAU SINGULARITY MANIFOLD FOR THE GRAPH $\mathcal{G}_4$

The momentum variables are assigned as shown in Fig. 3 in accordance with the prescription given above. It is well known that the p-space function corresponding to the graph $\mathcal{G}_n$ is given by an integral of a product of $\frac{1}{2}n(n-1)$ propagators integrated over a set of $\frac{1}{2}(n-1)(n-2)$ independent loop momenta. This last number is also equal to the number of internal lines not connected to any one vertex.

$$H_4(q_kq_l;m_{ij}^2)=\int\cdots\int\prod_{i<j}dp_{ij}\frac{1}{p_{ij}^2+m_{ij}^2}\prod_{k=1}^{3}\delta(q_k-\sum_{i\neq k}p_{ik}),\quad(17)$$

where q_k are a set of three independent external momenta of $\mathcal{G}_4$.

As usual, a set of Feynman parameters α_{ij}, $i\neq j=0, 1, 2, 3$, $\alpha_{ji}=\alpha_{ij}$, can be introduced and normalized to $\sum\alpha_{ij}=1$. Since we are not concerned here with the ultraviolet divergence problem, we assume that Eq. (17) is understood in the formal sense, and whenever necessary, an appropriate number of operations with $\sum\partial/\partial m_{ij}^2$ may be applied to render the resulting expression meaningful. As far as the analyticity over the external variables $(q_k\cdot q_l)$ are concerned, this operation does not change the essential structure of the singularity manifold.

It is well known that the singularity manifold of the function H_4 in (17) is embodied in the following set of algebraic equations of Landau.[17]

(a) *Momentum conservation at each vertex.* (The number of independent vertex is $n-1$.)

$$q_k=\sum_{i\neq k}p_{ik},\quad k=1,2,3;\, i=0,\cdots,3.\qquad(18)$$

Note that q_0 has been eliminated from the problem by virtue of

$$\sum_{i=0}^{3}q_i=0.\qquad(19)$$

(b) *Loop equations.* [The number of independent loops for $\mathcal{G}_n$ is $\frac{1}{2}(n-1)(n-2)$.]

$$\sum_{\substack{\text{closed}\\\text{path}}}\alpha_{ij}p_{ij}=0.\qquad(20)$$

(c) *Internal mass shell.* (for each internal line)

$$p_{ij}^2+m_{ij}^2=0,\quad i\neq j=0,\cdots,3.\qquad(21)$$

The above set of equations implicitly defines the singularity manifold in the inner product space of

$$z_{kl}=-(q_k\cdot q_l)\,.\qquad(22)$$

We want to show that the perturbation singularity manifold given by the solutions to the Landau equations can be parameterized in a form which is very similar to the Källén $\Xi(t)$ manifold.

From Eqs. (18) and (20), one can readily solve for q_k. Writing in the matrix form, we get

$$q=B\alpha p,\qquad(23)$$

where

$$q=\begin{bmatrix}q_1\\q_2\\q_3\end{bmatrix},\quad p=\begin{bmatrix}p_1\\p_2\\p_3\end{bmatrix}\equiv\begin{bmatrix}p_{01}\\p_{02}\\p_{03}\end{bmatrix},\qquad(24)$$

$$\alpha=\begin{bmatrix}\alpha_{01}&0&0\\0&\alpha_{02}&0\\0&0&\alpha_{03}\end{bmatrix},\qquad(25)$$

and

$$B=\begin{bmatrix}\beta_{01}+\beta_{12}+\beta_{13}&-\beta_{12}&-\beta_{13}\\-\beta_{12}&\beta_{02}+\beta_{12}+\beta_{23}&-\beta_{23}\\-\beta_{13}&-\beta_{23}&\beta_{03}+\beta_{13}+\beta_{23}\end{bmatrix},\qquad(26)$$

with

$$\beta_{ij}=\alpha_{ij}^{-1}.\qquad(27)$$

Note that B is a *symmetric* matrix. Equation (23) is the *vector solution* to the Landau equations. The inner products of p_k are now to be selected in "lengths"

[17] L. Landau, Zh. Eksperim. i Teor. Phys. **37**, 62 (1959) [English Transl.: Soviet Phys.—JETP **10**, 45 (1960)].

according to (21). Going into the space of invariants, we form the Gram matrix of (23), and get

$$Z=B\hat{a}B\,, \tag{28}$$

where

$$Z=\|z_{kl}\|\,,\quad z_{kl}=-(q_k\cdot q_l)\,, \tag{29}$$

$$\hat{a}=\|\hat{a}_{kl}\|\,,\quad \hat{a}_{kl}=-(\hat{p}_k\cdot\hat{p}_l)\,, \tag{30}$$

$$\hat{p}_k=\alpha_{0k}p_k. \tag{31}$$

The set of mass parameters $\hat{a}_{kl}$ is related to m_{ij}^2 as follows:

$$\hat{a}_{kk}=\alpha_{0k}^2m_{0k}^2,\quad k=1,2,3, \tag{32}$$

$$\hat{a}_{kl}=-\tfrac{1}{2}(\alpha_{kl}^2m_{kl}^2-\alpha_{0k}^2m_{0k}^2-\alpha_{0l}^2m_{0l}^2)\,;\quad k\neq l\neq 0. \tag{33}$$

Equations (28) gives the parametrized form of the perturbation singularity manifold for $\mathcal{G}_4$. For convenience, we shall refer to it as the Landau singularity manifold $\mathcal{L}_n(z;\hat{a})=0$.

5. DETERMINATION OF BOUNDARY OF THE LANDAU SINGULARITY MANIFOLD $\mathcal{L}_n(z;\hat{a})$

One immediately recognizes that the Landau singularity manifold $\mathcal{L}_n(z;\hat{a})$ for $\mathcal{G}_n$ in (28) is precisely of the same structure as the $\Xi_n(t)$ manifold in (15a). There are, however, the following modifications. While the a_{ik} in (15a) are real, the $\hat{a}_{ik}$ in (28) are, in general, complex, since for complex Z's, the Feynman parameters α_{ij} will be complex on the Landau manifold. This implies that (28) be regarded as an extension of (15a) into the complex a space. However, the problem of the determination of the geometric envelopes of the manifold (28), or (15a), is purely algebraic. The geometric envelope conditions for $\mathcal{L}_n(z;\hat{a})$ over the real m_{ij}^2 is trivially translated into the "envelope" conditions over the complex $\hat{a}_{ij}$. For example,

$$\frac{\partial\mathcal{L}}{\partial m_{0k}^2}=\alpha_{0k}^2\left(\frac{\partial\mathcal{L}}{\partial\hat{a}_{kk}}+\tfrac{1}{2}\sum_{l>k}\frac{\partial\mathcal{L}}{\partial\hat{a}_{kl}}\right), \tag{34a}$$

$$\frac{\partial\mathcal{L}}{\partial m_{kl}^2}=-\tfrac{1}{2}\alpha_{kl}^2\frac{\partial\mathcal{L}}{\partial\hat{a}_{kl}}\,, \tag{34b}$$

with the slopes of $\mathcal{L}_n(z;\hat{a})$ formally given by

$$-\partial\mathcal{L}/\partial t=\omega, \tag{35a}$$

$$\partial\mathcal{L}/\partial\hat{a}_{ik}=\tfrac{1}{2}\omega B_{ik}. \tag{35b}$$

It follows then that the geometric envelope conditions on m_{ij}^2 imply the reality conditions on the B_{ik}'s and the α's. In particular, Källén's result on the leading boundary of the $\Xi_n(t)$ manifold can now be taken over word for word. So the leading boundary of the Landau manifold $\mathcal{L}_n(z;\hat{a})$ is given by the geometric envelopes over the *nondiagonal masses*. By the prescription stated in (16), this means that

(i) the diagonal mass $\hat{a}_{kk}=0$, $k=1,\cdots,(n-1)$; and the diagonal B_{kk} are complex; (36a)

(ii) the off-diagonal mass $\hat{a}_{kl}\neq 0$ and the corresponding B_{kl} are *real*. (36b)

From (33), we now have

$$\hat{a}_{kl}=-\tfrac{1}{2}\alpha_{kl}^2m_{kl}^2<0. \tag{37}$$

With this substitution into (28), we get the leading boundary to $\mathcal{L}_n(z;\hat{a})$:

$$Z=-B\Lambda B, \tag{38}$$

where

$$\Lambda_{ik}=-\hat{a}_{ik}(1-\delta_{ik})\geqslant 0 \tag{39}$$

and the reality structure of B is specified in (36). In particular, for $n=4$, (38) can be written as the following 3×3 matrices.

$$Z=-\mathrm{D}\tilde{\mathrm{A}}\mathrm{N}\tilde{\mathrm{A}}\mathrm{D}, \tag{40}$$

where

$$N=D\Lambda D=\|(1-\delta_{ik})\| \tag{41a}$$

$$\tilde{\mathrm{A}}=D^{-1}BD^{-1} \tag{41b}$$

$$D_{jk}=(\Lambda_{jk}\Lambda_{jl}\Lambda_{kl}^{-1})^{-\frac{1}{2}}\delta_{jk},\quad j\neq k\neq l=1,2,3\text{ cyclic.} \tag{41c}$$

It is clear that the matrices D, $\tilde{\mathrm{A}}$, N in (40) have the canonical form (6). The manifold (40) will be referred to as the (DANAD)′ manifold. For the rank-2 case, it gives, of course, the well-known result of the F_{kl}' surface (2). The tilde sign on $\tilde{\mathrm{A}}$ in (40) is a reminder of the following: The canonical form of A in (6) does not specify the signs of A_{kl} versus, e.g., ReA_{kk}. Thus one cannot preclude the possibility that the relative signs for $\tilde{\mathrm{A}}_{kl}$ in the (DANAD)′ could be different from the corresponding signs of the A_{kl} in the original primitive boundary DANAD. In fact, there are reasons to believe that such a distinction of signs should exist.[18] Hence the tilde on $\tilde{\mathrm{A}}$ in (40).

6. CONCLUDING REMARKS

Several remarks are perhaps in order:

(1) It should be noted that in the present paper, we are only interested in the formal structure of the singularity manifold and its boundary. The question of relevance criteria, namely, which piece of the boundary is relevant for which specific configuration of the z_{kl}'s, is left entirely untouched here. The specific parametrization we have adopted in (23) or (28) has its merit on the grounds that techniques have already been sufficiently developed in the study of the boundary of the primitive domain D_4^+ to enable handling of manifolds

[18] This is partially a relevancy argument: Writing out $Z=\mathrm{DANAD}$, we have $z_{11}=2d_1^2[A_{11}(A_{12}+A_{13})+A_{12}A_{13}]$, etc. Since a change of relevance generally occurs at the intersection with the lower rank manifold, setting $A_{13}=0$ gives $z_{11}=2d_1^2A_{11}A_{12}$. Now the relevant part of the 3-point function boundary comes essentially when Re $z_{11}>0$. This implies A_{12} Re $A_{11}>0$ for the relevant part of DANAD. A similar computation for the $Z=-\mathrm{D}\tilde{\mathrm{A}}\mathrm{N}\tilde{\mathrm{A}}\mathrm{D}$ shows that the requirement is now $\tilde{A}_{12}(\mathrm{Re}\,\tilde{A}_{11})<0$. Furthermore, an inspection on the explicit form of B in (26) also suggests $\tilde{A}_{kl}(\mathrm{Re}\,\tilde{A}_{kk})<0$.

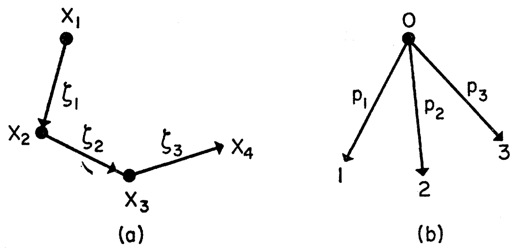

FIG. 4. A kind of duality: (a) chain-like vectors in x space; (b) cone-like vectors in p space.

precisely of the DANAD variety.[7,19] The fact that the perturbation boundary turns out exactly of the $Z=-\mathrm{D}\tilde{\mathrm{A}}\mathrm{N}\tilde{\mathrm{A}}\mathrm{D}$ form has rendered it unnecessary to attack the Landau equations by an explicit process of elimination of the Feynman parameters α_{ij}. In fact, a brute-force approach to graphs like $\mathcal{G}_4$ would be quite hopeless.

(2) Formally, it might be of interest to note that graphs with internal vertices may be regarded as degenerate cases of $\{\mathcal{G}_n\}$ when one or more external momenta are allowed to shrink to zero. For example, the Mercedes diagram considered by Källén and Toll[20] may be looked upon as the limit of the graph $\mathcal{G}_4$, e.g., as $q_3\to 0$.

(3) The situation for the p-space domain itself in the nonperturbative approach is much less transparent. While it is known that, for $n=3$, the p-space domain without mass spectrum coincides with the x-space domain,[3] in general this is not the case[21] (x-space domain is larger). A more modest question would be to determine the domain of the graph $\mathcal{G}_4$ *with* the inclusion of mass spectral conditions. Such a problem for the 3-point function has been studied by W. Brown.[22]

(4) Finally, we attempt to give a geometrical picture of the underlying difference between the x-space DANAD and the p-space (DANAD)′. For the original x-space primitive domain, one works with the (unpermuted) vacuum expectation values (3).

On the x-space DANAD, one has[6,9]

$$\zeta=DA\eta, \tag{42}$$

where $\zeta_k=\xi_k-i\eta_k$, η_k are light-like on the boundary and (for rank less than 4), $-(\eta_k\cdot\eta_l)$, $(k\neq l)$ may be normalized to 1 by absorbing the scale factors in D and A [cf. Eq. (41)]. Thus $-(\eta_k\cdot\eta_l)$ is the matrix N of (7). Since the ζ_k are *difference* vectors between consecutive points x_k's, both ξ_k and η_k may be regarded, so to speak, as nearest-neighbor (or *chain-like*) displacements. Then, obviously, the displacements between the next-nearest neighbors are given as the *sum* of nearest-neighbor displacements. See Fig. 4(a). Thus, for example, one has

$$-(\eta_k\cdot\eta_l)=\tfrac{1}{2}[-(\eta_k+\eta_l)^2+\eta_k^2+\eta_l^2]\,. \tag{43}$$

On the other hand, for the p-space perturbation domain, the set of vectors p_k (i.e., p_{0k}) that comes into play has the *cone-like* structure, namely all vectors being projected from a common vertex; see Fig. 4(b). This implies that every third vector is given as the *difference* of two vectors, e.g.,

$$-(p_k\cdot p_l)=-\tfrac{1}{2}[-(p_k-p_l)^2+p_k^2+p_l^2]\,. \tag{44}$$

Thus there is a net difference in sign between the off-diagonal elements (of the matrix N, practically) on the right-hand sides of (43) and (44). One might thus visualize the situation as follows:

(i) The x-space DANAD is defined with respect to a set of chain-like vectors η_k: boundary of primitive domain.

(ii) The p-space DANAD is defined with respect to a set of cone-like vectors p_k: boundary of perturbation domain without mass spectral condition.

(iii) (DANAD)′ is the result of translating p-space DANAD back into the canonical language of the x-space DANAD, namely with D, A, N all in their canonical form. We have seen that (DANAD)′ should read as $Z=-\mathrm{D}\tilde{\mathrm{A}}\mathrm{N}\tilde{\mathrm{A}}\mathrm{D}$.

If one visualizes the postulate of local commutativity as having the power to reverse the links in the primitive chain, then it would be extremely interesting, should the following conjecture be true[23]: The envelope of holomorphy of the union of permuted domains (each being bounded by a DANAD variety corresponding to different sets of chain-like vectors) is a domain primarily[4] bounded by the cone-like DANAD. The result of this paper shows that the cone-like DANAD, viz., the (DANAD)′, indeed defines a natural domain of holomorphy.

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APPENDIX A

The function $\Xi_n(t)$ defined in Eq. (11) may be written more explicitly as follows: For convenience, write $t=\sigma_0^{\frac{1}{2}}$.

Case (a) $n=2$ (trivial)

$$\Xi_2(\sigma)=\begin{vmatrix}\sigma_0^{\frac{1}{2}} & \sigma_1^{\frac{1}{2}}\\ \sigma_1^{\frac{1}{2}} & \sigma_0^{\frac{1}{2}}\end{vmatrix} \tag{A1}$$

Case (b) $n=3$

$$\Xi_3(\sigma)=\lambda(\sigma_0,\sigma_1,\sigma_2)\,, \tag{A2}$$

[19] J. S. Toll, in *Lectures on Field Theory and Many-Body Problem* (Academic Press Inc., New York, 1961), p. 147.
[20] G. Källén and J. S. Toll, Helv. Phys. Acta **33**, 753 (1960).
[21] See references cited on p. 448 of Ref. 6.
[22] W. S. Brown, Ph.D. thesis, Princeton, 1961 (unpublished); J. Math. Phys. **3**, 221 (1962).
[23] Cf. R. F. Streater, Nuovo Cimento **15**, 937 (1960).

where

$$\lambda(\alpha,\beta,\gamma)\equiv\alpha^2+\beta^2+\gamma^2-2\alpha\beta-2\beta\gamma-2\gamma\alpha\,. \quad \text{(A3)}$$

Geometrically, one may visualize that in the $\sigma^{\frac{1}{2}}$ space

$$-\lambda(\sigma_0,\sigma_1,\sigma_2)=16\Delta^2(\sigma_0^{\frac{1}{2}},\sigma_1^{\frac{1}{2}},\sigma_2^{\frac{1}{2}})\,, \quad \text{(A4)}$$

where $\Delta(a,b,c)$ denotes the area of a triangle with side lengths a,b,c. In this picture, the vanishing of (A2) implies that the triangle $(\sigma_0^{\frac{1}{2}},\sigma_1^{\frac{1}{2}},\sigma_2^{\frac{1}{2}})$ collapses. There are two possibilities to collapse the triangle: (i) One vertex of the triangle falls on the base; or (ii) one side $(\sigma_0^{\frac{1}{2}})$ of the triangle shrinks to zero. It can be easily verified that possibility (i) yields the F_{ij} surface and (ii) yields the S surface, for the boundary of the 3-point function primitive domain D_3^+.

Case (c) $n=4$

$$\Xi_4(\sigma)=K(\sigma_0,\sigma_k)\cdot K(-\sigma_0,\sigma_k)\,, \quad \text{(A5)}$$

where

$$K(\sigma)=\begin{vmatrix} \sigma_0^{\frac{1}{2}} & \sigma_1^{\frac{1}{2}} & \sigma_2^{\frac{1}{2}} & \sigma_3^{\frac{1}{2}} \\ \sigma_1^{\frac{1}{2}} & \sigma_0^{\frac{1}{2}} & \sigma_3^{\frac{1}{2}} & \sigma_2^{\frac{1}{2}} \\ \sigma_2^{\frac{1}{2}} & \sigma_3^{\frac{1}{2}} & \sigma_0^{\frac{1}{2}} & \sigma_1^{\frac{1}{2}} \\ \sigma_3^{\frac{1}{2}} & \sigma_2^{\frac{1}{2}} & \sigma_1^{\frac{1}{2}} & \sigma_0^{\frac{1}{2}} \end{vmatrix}. \quad \text{(A6)}$$

Geometrically, again in the $\sigma^{\frac{1}{2}}$ space, we have

$$16\square^2=-K(-\sigma_0,\sigma_k)\,, \quad \text{(A7)}$$

where $\square$ denotes the area of a cyclic quadrilateral[24] with side lengths $(\sigma_0,\sigma_1,\sigma_2,\sigma_3)$. It is evident that

$$\Xi_3(\sigma_0,\sigma_1,\sigma_2)=K(\sigma_0,\sigma_1\sigma_2,0)=[\Xi_4(\sigma_0,\sigma_1,\sigma_2,0)]^{\frac{1}{2}} \quad \text{(A8)}$$

Case (d) $n=5$

Contrary to intuition, $\Xi_5(\sigma)$ is not expressible in terms of 5×5 determinants. Instead, it will be degenerate cases of 8×8 determinants.

[24] See, e.g., G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, New York, 1944), 2nd ed. p. 414. A cyclic quadrilateral is one whose four vertices lie on a circle; given the lengths of four sides, it corresponds to the quadrilateral with maximum area. I wish to thank T. T. Wu for a remark on this last statement.

$$\Xi_5(\sigma_0,\cdots,\sigma_4)=[\Xi_8(\sigma_0,\cdots,\sigma_4,0,0,0)]^{1/8} = \Omega_1\Omega_2|_{\sigma_5=\sigma_6=\sigma_7=0} \quad \text{(A9)}$$

and

$$\Xi_8(\sigma)=\prod_{k=1}^{16}\Omega_k(\sigma) \quad \text{(A10)}$$

where[25]

$$\Omega_1(\sigma)=\begin{vmatrix} \sigma_0^{\frac{1}{2}} & \sigma_1^{\frac{1}{2}} & \sigma_2^{\frac{1}{2}} & \sigma_3^{\frac{1}{2}} & \sigma_4^{\frac{1}{2}} & \sigma_5^{\frac{1}{2}} & \sigma_6^{\frac{1}{2}} & \sigma_7^{\frac{1}{2}} \\ \sigma_1^{\frac{1}{2}} & \sigma_0^{\frac{1}{2}} & \sigma_3^{\frac{1}{2}} & \sigma_2^{\frac{1}{2}} & \sigma_5^{\frac{1}{2}} & \sigma_4^{\frac{1}{2}} & \sigma_7^{\frac{1}{2}} & \sigma_6^{\frac{1}{2}} \\ \sigma_2^{\frac{1}{2}} & \sigma_3^{\frac{1}{2}} & \sigma_0^{\frac{1}{2}} & \sigma_1^{\frac{1}{2}} & \sigma_6^{\frac{1}{2}} & \sigma_7^{\frac{1}{2}} & \sigma_4^{\frac{1}{2}} & \sigma_5^{\frac{1}{2}} \\ \sigma_3^{\frac{1}{2}} & \sigma_2^{\frac{1}{2}} & \sigma_1^{\frac{1}{2}} & \sigma_0^{\frac{1}{2}} & \sigma_7^{\frac{1}{2}} & \sigma_6^{\frac{1}{2}} & \sigma_5^{\frac{1}{2}} & \sigma_4^{\frac{1}{2}} \\ \sigma_4^{\frac{1}{2}} & \sigma_5^{\frac{1}{2}} & \sigma_6^{\frac{1}{2}} & \sigma_7^{\frac{1}{2}} & \sigma_0^{\frac{1}{2}} & \sigma_1^{\frac{1}{2}} & \sigma_2^{\frac{1}{2}} & \sigma_3^{\frac{1}{2}} \\ \sigma_5^{\frac{1}{2}} & \sigma_4^{\frac{1}{2}} & \sigma_7^{\frac{1}{2}} & \sigma_6^{\frac{1}{2}} & \sigma_1^{\frac{1}{2}} & \sigma_0^{\frac{1}{2}} & \sigma_3^{\frac{1}{2}} & \sigma_2^{\frac{1}{2}} \\ \sigma_6^{\frac{1}{2}} & \sigma_7^{\frac{1}{2}} & \sigma_4^{\frac{1}{2}} & \sigma_5^{\frac{1}{2}} & \sigma_2^{\frac{1}{2}} & \sigma_3^{\frac{1}{2}} & \sigma_0^{\frac{1}{2}} & \sigma_1^{\frac{1}{2}} \\ \sigma_7^{\frac{1}{2}} & \sigma_6^{\frac{1}{2}} & \sigma_5^{\frac{1}{2}} & \sigma_4^{\frac{1}{2}} & \sigma_3^{\frac{1}{2}} & \sigma_2^{\frac{1}{2}} & \sigma_1^{\frac{1}{2}} & \sigma_0^{\frac{1}{2}} \end{vmatrix}. \quad \text{(A11)}$$

The next 8 Ω's, namely $\Omega_2,\cdots,\Omega_9$, are gotten by inverting the sign of one single $\sigma_i^{\frac{1}{2}}\to-\sigma_i^{\frac{1}{2}}$, from $\Omega_1(\sigma)$, $i=0,\cdots,7$. Finally, the remaining 7 Ω's, namely $\Omega_{10},\cdots,\Omega_{16}$ are gotten by inverting the signs of a pair of $\sigma^{\frac{1}{2}}$'s from Ω_1, specifically $\sigma_0^{\frac{1}{2}}\to-\sigma_0^{\frac{1}{2}}$ together with one additional $\sigma_k^{\frac{1}{2}}\to-\sigma_k^{\frac{1}{2}}$ at a time, $(k\neq0)$. This completes the characterization for $\Xi_8(\sigma)$.

With (A10) for $n=8$, all previous expressions for $\Xi_n(\sigma)$ for $n<8$ may now be regarded as degenerate cases of (A10) by inserting appropriate zero entries in (A10).

The precise geometrical meaning for the quantities like (A11) for rank higher than 4 is, unfortunately, not known to the present author.

[25] Clearly, each $\Omega_k(\sigma)$ contains eight factors of $(\Sigma\pm\sigma_k^{\frac{1}{2}})$. Symbolically, we have for Ω_1

$$\Omega_1=(\text{all}+)(0123+)(0145+)(0167+) \times(0246+)(0257+)(0347+)(0356+),$$

where in the last seven factors, the eight $\pm\sigma_k^{\frac{1}{2}}$'s break up into four +'s and four −'s and we have written out four of the indices which have the plus signs. One notes here that the strange combination of those indices in triplets (besides 0) happens to coincide with the multiplication table of the octonian algebra. See, e.g., A. Pais, Phys. Rev. Letters **7**, 291 (1961). It is not clear whether there is anything deep about such objects as the 8×8 determinant Ω_1. Our Ω_2 is gotten from Ω_1 by letting $\sigma_0^{\frac{1}{2}}\to-\sigma_0^{\frac{1}{2}}$, a discrete operation which is also discussed by Pais in an entirely different context.